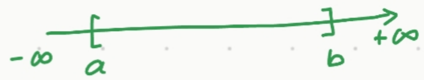
June 20^{th}, 2024

Lecture 17

§7.8 Improper integrals

$\int_a^b f(x)\,dx \rightarrow$ f is a continuous function defined on the interval $[a, b]$

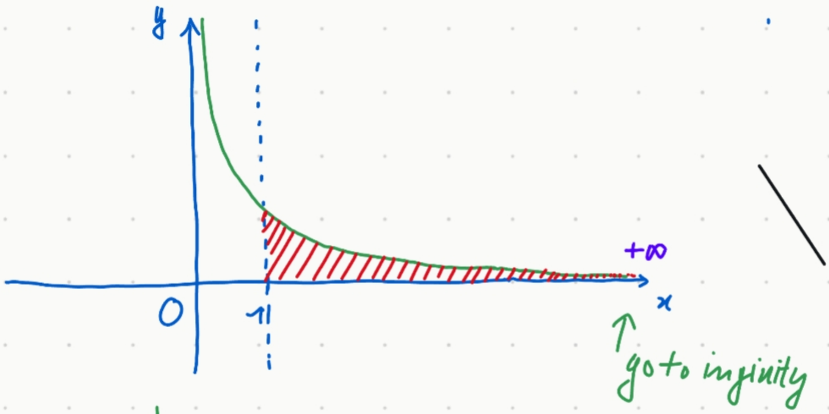


Improper integrals

1) $[a, b]$ infinite interval
   $[a, +\infty)$, $(-\infty, b]$, $(-\infty, +\infty)$
2) f has an infinite discontinuity in $[a, b]$

Type 1: Infinite intervals

Ex: Consider the region under the curve $y = \frac{1}{x^2}$ above x-axis, and to the right of the line $x = 1$.



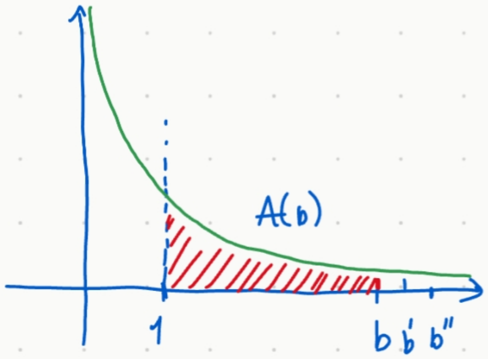
$\Rightarrow$ This region is unbounded.

Task: Compute the area of this region

Task: Compute the area of this region:

We know how to compute the area of a bounded region

Assume it is a bounded region:



Area from 1 to b $(A(b))$

b'	$A(b')$
b''	$A(b'')$
$\vdots$	$\vdots$
$+\infty$	$A(+\infty)$

⇒ Find the area depending on b

⇒ Take the limit when b tends to $+\infty$.

choose $b > 1$

$$A(b) = \int_1^b \frac{1}{x^2}\,dx = \int_1^b x^{-2}\,dx = -x^{-1}\Big|_1^b$$

$$= -\frac{1}{b} - \frac{-1}{1}$$

$$A(b) = 1 - \frac{1}{b}$$

$$A(+\infty) = \lim_{b\to+\infty} A(b) = \lim_{b\to+\infty}\left(1 - \frac{1}{b}\right) = 1$$

$\frac{1}{+\infty} = 0.$

Definition Improper integral of type 1

1) $\int_a^t f(x)\,dx$ exists for every $t \geq a$

then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

Derive similarly for

for every $t \geq a$

then $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

Improper integral.

Define similarly for $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$

The improper integrals $\displaystyle\int_a^{\infty} f(x)\,dx$, $\displaystyle\int_{-\infty}^{b} f(x)\,dx$ are called

convergent if the limit exists and
divergent if the limit does not exists.

If both $\displaystyle\int_a^{+\infty} f(x)\,dx$, $\displaystyle\int_{-\infty}^{a} f(x)\,dx$ are convergent,

then $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx$.

Example: Determine whether the integral

$$\int_1^{\infty} 1/x \, dx$$

is convergent or divergent.

By definition, we have

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to+\infty} \ln|x| \Big|_1^b$$

$$= \lim \ \ln b$$

$$= \lim \ln|x| \Big|_1$$

$$= \lim_{b \to +\infty} \ln b$$

$$= +\infty$$

$\Rightarrow$ The limit does not exist as a finite number so the improper integral

$$\int_1^\infty \frac{1}{x}\,dx$$

is divergent. $\boxtimes$

From the above examples:

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges} \qquad \int_1^\infty \frac{1}{x}\,dx \text{ diverges.}$$

Example: Evaluate $\int_{-\infty}^0 x e^x\,dx$.

By definition $\int_{-\infty}^0 x e^x dx = \lim_{t \to -\infty} \int_t^0 x e^x\,dx$.

Compute $\int_t^0 x e^x\,dx$

Integration by parts:

$$\begin{cases} u = x \\ dv = e^x\,dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = e^x \end{cases}$$

So $$\int_t^0 x e^x dx = x e^x \Big|_t^0 - \int_t^0 e^x\,dx$$

$$= (0.e^0 - te^t) - e^x \Big|_t^0$$

$$= -te^t - (e^0 - e^t)$$

$1 + e^t$

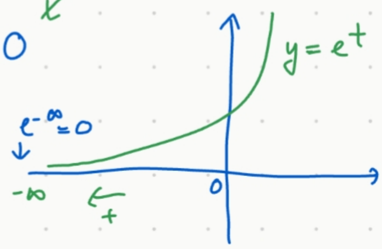
Thus $\displaystyle\int_{-\infty}^{0} x e^x\,dx = \lim_{t\to\infty} \left(-te^t - 1 + e^t\right)$

$\displaystyle = \left(\lim_{t\to-\infty}(-te^t)\right) - \left(\lim_{t\to-\infty} 1\right) + \left(\lim_{t\to-\infty} e^t\right)$

$(e^{-\infty} = 0)$

$= 0 - 1 + 0$

$= -1$



$\displaystyle\Rightarrow \int_{-\infty}^{0} x e^x\,dx$ converges

Example: Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$.

By definition $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \underline{\int_{0}^{+\infty} \frac{1}{1+x^2}\,dx}$

Compute $\displaystyle\int_{0}^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$

$\displaystyle = \lim_{t\to\infty} \tan^{-1} x \Big|_0^t$

$\displaystyle = \lim_{t\to\infty} \left(\tan^{-1} t - \underbrace{\tan^{-1} 0}_{=0}\right)$

$\displaystyle = \lim_{t\to\infty} \tan^{-1} t. = \lim_{t\to+\infty} \arctan(t)$

$\displaystyle = \frac{\pi}{2}.$

because $\displaystyle \tan x = \frac{\sin x}{\cos x.} = \infty$

means that $\sin x > 0$, $\cos x = 0$

$\arctan(\tan(y)) = y$

arctan...

$\lim^{-1} t. = \lim_{t \to +\infty} \overset{0}{\text{arcta}}$

$= \frac{\pi}{2}.$

because $\tan x = \frac{\sin x}{\cos x} = \infty$

means that $\sin x > 0$, $\cos x = 0$

$\arctan(\tan(y)) = y$

$\arctan(t) = ?$

set $t = \tan(y)$ find y.

In our problem $\arctan(\infty)$ so we set

$\infty = \tan(y)$

find y. $\Rightarrow$ $y = \frac{\pi}{2}$.

Another example $\arctan(1) = \arctan(\tan(y))$

and find y.

we set $1 = \tan y \Rightarrow y = \frac{\pi}{4}$.

$\Rightarrow \arctan(1) = \frac{\pi}{4}$.

We saw that

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}, \quad \int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_0^\infty \frac{1}{1+x^2}\,dx + \int_{-\infty}^0 \frac{1}{1+x^2}\,dx$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

conclusion: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converges.

Example. For what values of p the integral
$$\int_1^\infty \frac{1}{x^p}\,dx$$
convergent?

We know $\int_1^\infty \frac{1}{x^2}\,dx < \infty$, $\int_1^\infty \frac{1}{x}\,dx$ divergen

Assume $p \neq 1$, we have

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$$
$$= \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$
$$= \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^t$$
$$= \lim_{t\to\infty} \frac{1}{1-p}\left(t^{-p+1} - 1^{-p+1}\right)$$
$$= \lim_{t\to\infty} \frac{1}{1-p}\cdot t^{1-p} - \frac{1}{1-p}$$

Since $p \neq 1$, we have two cases.

- $p > 1 \Rightarrow 1-p < 0$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}\cdot t^{1-p} - \frac{1}{1-p} = -\frac{1}{1-p}$$

$1-p < 0 \Rightarrow t^{1-p} \to 0$

So the integral converges

- when $p < 1 \Rightarrow p-1 < 0$

$\Rightarrow t^{1-p} \to \infty$ as $t \to \infty$

So $$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}\,t^{1-p} - \frac{1}{1-p} \to \infty$$

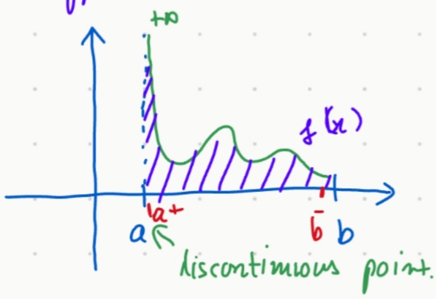
The int... diverges

$\dots - \frac{1}{1-p} \to \infty$

the integral diverges.

Conclusion: $\int_1^{\infty} \frac{1}{x^p}\,dx$ is convergent if $p > 1$
divergent if $p \le 1$.

II. Type 2: Discontinuous integrands.



We consider a finite interval $[a,b]$. f has discontinuous point at a or b.

⇒ Compute the area of shaded region under the graph of $f(x)$

Definition.

a) If f is continuous on $[a,b)$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

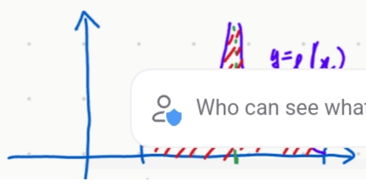
b)
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

The integral $\int_a^b f(x)$ is called convergent if the corresponding limit exists and divergent if the limit does not exists.

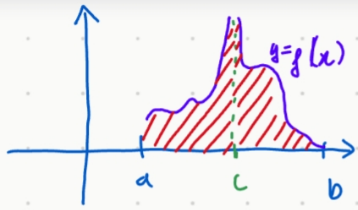
c) If f has discontinuity at c: $a < c < b$

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

if these integrals are convergent.



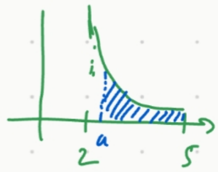
...ls are convergent.



Example: Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

The function $f(x) = \frac{1}{\sqrt{x-2}}$ is not defined at $x=2$

or f is discontinuous at $x=2$



By defimtion $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \underline{\lim_{a\to 2^+} \int_a^5 \frac{1}{\sqrt{x-2}}\,dx}$

$\leftarrow$ limit when $a\to 2^+$

Substitution: Put $u = x-2 \Rightarrow du = dx$.

$\begin{cases} x=5 \\ x=a \end{cases} \to \begin{cases} u=3 \\ u=a-2 \end{cases}$

$$\int_a^5 \frac{1}{\sqrt{x-2}}\,dx = \int_{a-2}^3 \frac{1}{\sqrt{u}}\,du = \frac{u^{1/2}}{1/2}\Big|_{a-2}^3$$

$$= 2u^{1/2}\Big|_{a-2}^3$$

$$= 2(\sqrt{3} - \sqrt{a-2})$$

Thus $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{a\to 2^+} \int_a^5 \frac{1}{\sqrt{x-2}}\,dx$

$$= \lim_{a\to 2^+} 2(\sqrt{3} - \sqrt{a-2})$$

$$= 2(\sqrt{3} - 0)$$

$$= 2\sqrt{3}$$

Conclusion: The improper integral is convergent.

Example: Determine whether $\int_0^{\pi/2} \sec x\,dx$ converges or diverges

$\cos x$

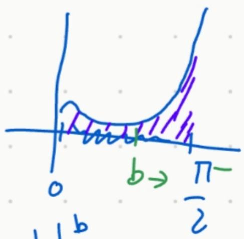
Example: Determine whether $\int_0 \sec x \, dx$

We know $\sec x = \dfrac{1}{\cos x} \to \infty$ as $x \to \frac{\pi}{2}$ (with $\cos x \to 0^+$)

so $f(x) = \sec x$ has discontinuity at $x = \frac{\pi}{2}$

$$\int_0^{\pi/2} \sec x \, dx = \int_{\to 0}^{\to \pi/2} \frac{1}{\cos x} \, dx$$

$$= \lim_{b \to \frac{\pi}{2}^-} \int_0^b \sec x \, dx$$



$$= \lim_{b \to \frac{\pi}{2}^-} \ln|\sec x + \tan x| \Big|_0^b$$

$$= \lim_{b \to \frac{\pi}{2}^-} \left(\ln|\sec b + \tan b| - \ln|\sec 0 + \tan 0| \right)$$

($\sec 0 = 1$, $\tan 0 = 0$; $= \ln|1| = 0.$)

$$\Rightarrow \int_0^{\pi/2} \sec x \, dx = \lim_{b \to \frac{\pi}{2}^-} \left(\ln|\sec b + \tan b| \right)$$

$$\int_0^{\pi/2} \sec x\,dx = \lim_{b\to \frac{\pi}{2}^-} \ln|\sec b + \tan b| \quad \sec b \to \infty$$

$$= \infty$$

Conclusion: The improper integral is divergent.

Example: Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

$f(x) = \frac{1}{x-1}$ has discontinuity at $x=1$

By definition, we have

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

Compute $\int_0^1 \frac{dx}{x-1} = \lim_{b\to 1^-} \int_0^b \frac{dx}{x-1}$

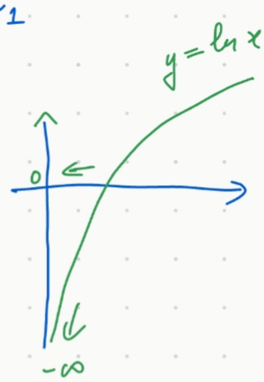
$$= \lim_{b\to 1^-} \ln|x-1| \Big|_0^b$$

$$= \lim_{b\to 1^-} \left(\ln|b-1| - \underbrace{\ln|0-1|}_{\ln|1|}\right)$$

$$= \lim_{b\to 1^-} \ln|b-1|$$

$$= \lim_{b\to 1^-} \ln(1-b)$$

$$= -\infty$$



Conclusion: $\int_0^1 \frac{1}{x-1}\,dx$ is divergent.

$$\Rightarrow \int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$$

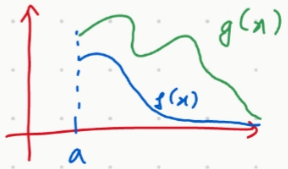
diverges diverges

diverges

* Comparison test for improper integrals

Comparison theorem

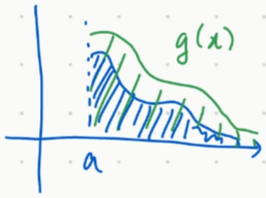
$0 \le f(x) \le g(x)$



1) $\int_a^\infty g(x)\,dx$ is convergent

$\Rightarrow \int_a^\infty f(x)\,dx$ is convergent

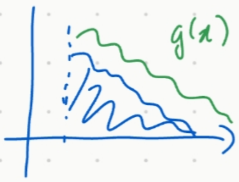
2) $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow \int_a^\infty g(x)\,dx$ is divergent

1) 

If green region has finite area

$\Rightarrow$ blue region has finite area.

$\int_a^\infty g(x)\,dx$ converges $\Rightarrow \int_a^\infty f(x)\,dx$ converges

2) 

If blue region has infinite area.

$\Rightarrow$ green region has infinite area

$\int_a^\infty f(x)\,dx$ diverges $\Rightarrow \int_a^\infty g(x)\,dx$ diverges

1 Example: Determine whether

$$\int_0^\infty e^{-x^2}\,dx$$ converges or diverges?

$f(x) = e^{-x^2}$, what is $g(x) = ?$

(some function

that we already

or converges)

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...nverges or diverges ?

$f(x) = e^{-x^2}$, what is $g(x) = ?$
(some function that we already knew that $\int g(x)$ diverges or converges)

For example $g(x) = \frac{1}{x}$, or $\frac{1}{x^2}$.

$\int_1^\infty \frac{1}{x}\,dx$ is divergent, $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent

$$\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\leq\ C} + \underbrace{\int_1^\infty e^{-x^2}dx}_{I}$$

because e^{-x^2} is continuous on $[0,1]$

$$I = \int_1^\infty \frac{1}{e^{x^2}}\,dx$$

on $[1, \infty]$ $\therefore$ $f(x) = \frac{1}{e^{x^2}} \leq \frac{1}{x^2} = g(x)$

$\leq \frac{1}{e^x} = g(x)$

Then: $\int_1^\infty \frac{1}{e^x}\,dx$ is converges

$\Rightarrow \int_1^\infty \frac{1}{e^{x^2}}\,dx$ is converges. by Comparison Test

Example: $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent

because $f(x) = \frac{1+e^{-x}}{x} > \frac{1}{x}$

and $\int_1^\infty \frac{1}{x}\,dx$ is divergent

By the Comparison Test $\Rightarrow \int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent.

Who can see what you share here 67%